# An Overview of Optimization

## History, Classification and Methods Part 1

MORSC Tutorial

**Carlos Armando Zetina**
Concordia University and CIRRELT

# Outline

- What is Optimization?
- History of Optimization
    - Pre Simplex Era
    - Modern Optimization and applications
- NEOS server

# What is optimization?

- an act, process, or methodology of making something (as a design, system, or decision) as fully perfect, functional, or effective as possible; specifically : the mathematical procedures (as finding the maximum of a function) involved in this.
  \- *Merriam Webster Dictionary.*

In Operations Research

**Optimization=Mathematical Optimization**

No Heuristics or Metaheuristics in this part.

# Mathematical Optimization

A Mathematical Optimization Problem:

$$\begin{aligned} (\mathrm{P}) \quad & \min h(x) & \\ \text{s.t.} \quad & f_t(x) \leq 0 & \forall t \in T \\ & g_s(x) = 0 & \forall s \in S \\ & x \in X & \end{aligned}$$

$h, f, g : X \to \mathbb{R}$

X is a finite dimensioned space

T,S are arbitrary index sets

Mathematical Optimization refers to the study of these problems' properties, development and implementation of solution algorithms and their application to real world problems.
*-Mathematical Optimization Society (INFORMS)*

# Mathematical Optimization

A Mathematical Optimization Problem:

$$\begin{array}{llll} (\mathrm{P}) & \min h(x) & & \text{Objective Function} \\ \text{s.t.} & f_t(x) \leq 0 & \forall t \in T & \text{Inequality Constraints} \\ & g_s(x) = 0 & \forall s \in S & \text{Equality Constraints} \\ & x \in X & & \text{Variable Definition} \end{array}$$

**Constraints**

Mathematical Optimization refers to the study of these problems' properties, development and implementation of solution algorithms and their application to real world problems.
*-Mathematical Optimization Society (INFORMS)*

# Some Applications of Optimization

## Finance

- Portfolio Management
- Asset Pricing
- Risk Management
- Financial Valuation

## Manufacturing

- Product Mix Planning
- Inventory Management
- Job Scheduling
- Maintenance Scheduling

## Health Care

- Staff Scheduling
- Radiation Exposure
- Blood Bank collection
- Ambulance Scheduling

## Transportation- Airlines

- Fleet assignment
- Crew Scheduling
- Fuel Allocation
- Airport Traffic Planning

## Transportation- Other

- Vehicle Routing
- Depot/Warehouse location
- Transportation System Man.
- Fleet Management

## Communications/Computing

- Circuit Design
- Office Automation
- Telephone Operator Scheduling

ETC…

# The History of Optimization

- Can be traced back as far as 300 BC
- Can be divided into two eras:

**Pre Simplex Era**

**1947**

**Modern Optimization**

Joseph Fourier

Pierre de Fermat

George Dantzig

Ralph Gomory

Photos from: Wikipedia.com

# The History of Optimization

## Pre Simplex Era

Joseph Fourier

Pierre de Fermat

Leonid Kantorovich

Tjalling Koopmans

Photos from: Wikipedia.com

# Pre Simplex Era

| | | |
|---|---|---|
| 300 BC | Euclid of Alexandria | • Minimal Distance b/w a point and a line<br>• Square is rectangle with max area given perimeter |
| 200 BC | Zenodorus | • Dido’s Problem- Semicircle |
| 1615 | Johannes Kepler | • Wine Barrel Problem<br>• Secretary Problem |
| 1636 | Pierre de Fermat | • Derivative vanishes at local optimum |
| 1660s | I. Newton, G. Leibniz | • Calculus of Variations |

# Pre Simplex Era

| | | |
|---|---|---|
| 1736 | Leonard Euler | • Foundations of Graph Theory |
| 1754 | Joseph Lagrange | • Maximum and Minimum w/ Constraints |
| 1823 | Joseph Fourier | • Rudimentary LP Programming algorithm |
| 1902 | Gyula Farkas | • Farkas Lemma |
| 1905 | Johan Jensen | • Convex Functions |
| 1917 | Harris Hancock | • Published Theory of Maxima and Minima |

# Pre Simplex Era

| | | |
|---|---|---|
| 1927 | Karl Menger | • Max number of p-q paths=Minimum Cut |
| 1931 | Dénes König | • Maximum matching= Minimum vertex cover in bipartite graphs |
| 1935 | Hassler Whitney | • Matroids |
| 1939 | Leonid Kantorovich | • Linear Programming Methods<br>• Transportation Problem |
| 1941 | Frank Hitchcock | • Transportation Problem |
| 1942 | Tjalling Koopmans | • Sensitivity Analysis<br>• Transportation Problem |

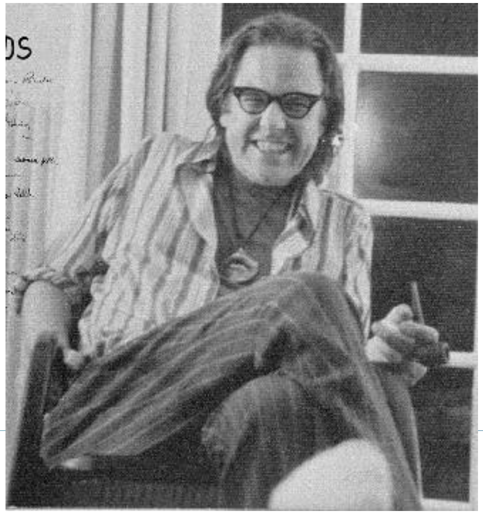
# The History of Optimization

## Modern Optimizaion

George Dantzig

Ralph Gomory

Martin Grötschel



Jack Edmonds

Photos from: Wikipedia.com, zib.de, Combinatorial Optimization-”Eureka you Shrink!

# The first problem of Modern Optimization

Given a set of Recommended Dietary Allowances and a list of 77 foods with their prices and nutrition content what is the cheapest survival diet possible?

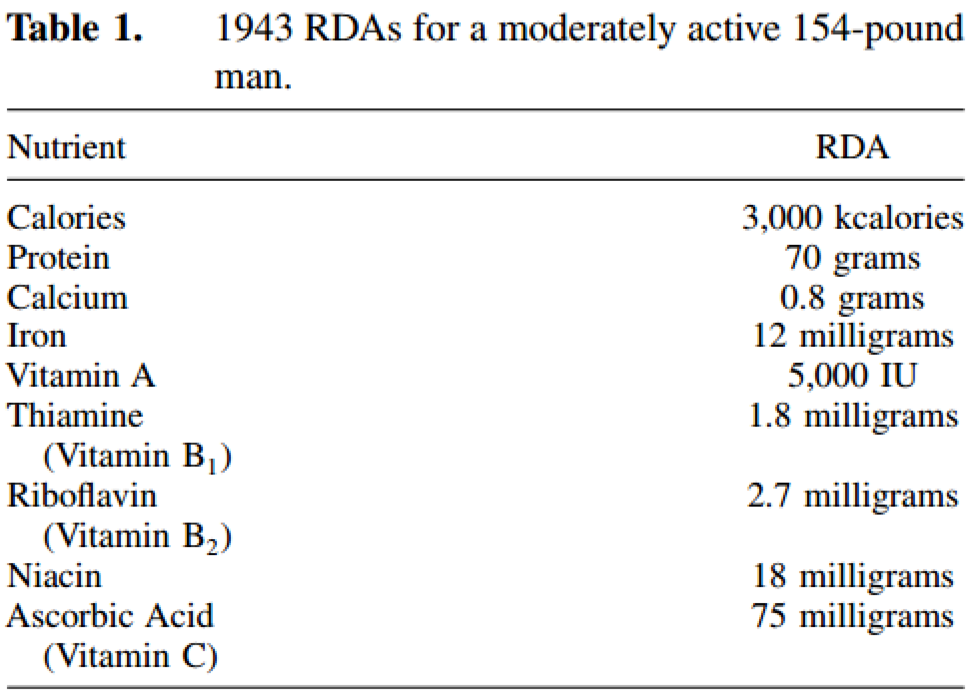


**Table 1.** 1943 RDAs for a moderately active 154-pound man.

| Nutrient | RDA |
|---|---|
| Calories | 3,000 kcalories |
| Protein | 70 grams |
| Calcium | 0.8 grams |
| Iron | 12 milligrams |
| Vitamin A | 5,000 IU |
| Thiamine (Vitamin $B_1$) | 1.8 milligrams |
| Riboflavin (Vitamin $B_2$) | 2.7 milligrams |
| Niacin | 18 milligrams |
| Ascorbic Acid (Vitamin C) | 75 milligrams |

# Linear Programming (Optimization)

## Stigler’s Diet (1943)

$$(\mathrm{P}) \qquad \min \sum_{j \in J} c_j x_j$$

$$\text{s.t.} \qquad \sum_{j \in J} a_{ij} x_j \geq RDA_i \qquad \forall i \in I$$

$$x_j \geq 0 \qquad \forall j \in J$$

*$J$ denote the set of 77 food items*

*$c_j$ denote the cost of food item $j \in J$.*

*$I$ denote the set of 9 essential nutrients in a diet*

*$RDA_i$ denote the amount of nutrient $i \in I$ necessary to survive.*

*$a_{ij}$ denote the amount of nutrient $i \in I$ contained in 1 unit of food item $j \in J$.*

*$x_j$ denote the amount of food item $j \in J$ to be consumed in the diet.*

# Linear Programming (Optimization)

## Stigler’s Diet (1943)

$$(\mathrm{P}) \qquad \min \sum_{j \in J} c_j x_j$$

$$\text{s.t.} \qquad \sum_{j \in J} a_{ij} x_j \geq RDA_i \qquad \forall i \in I$$

$$x_j \geq 0 \qquad \forall j \in J$$

## Linear Programming

**Objective Function- Linear**

**Constraints- Finite and Linear**

**Variable Definition- Continuous**

Solved by Trial and Error with Mathematical intuition

# Solving LPs Today

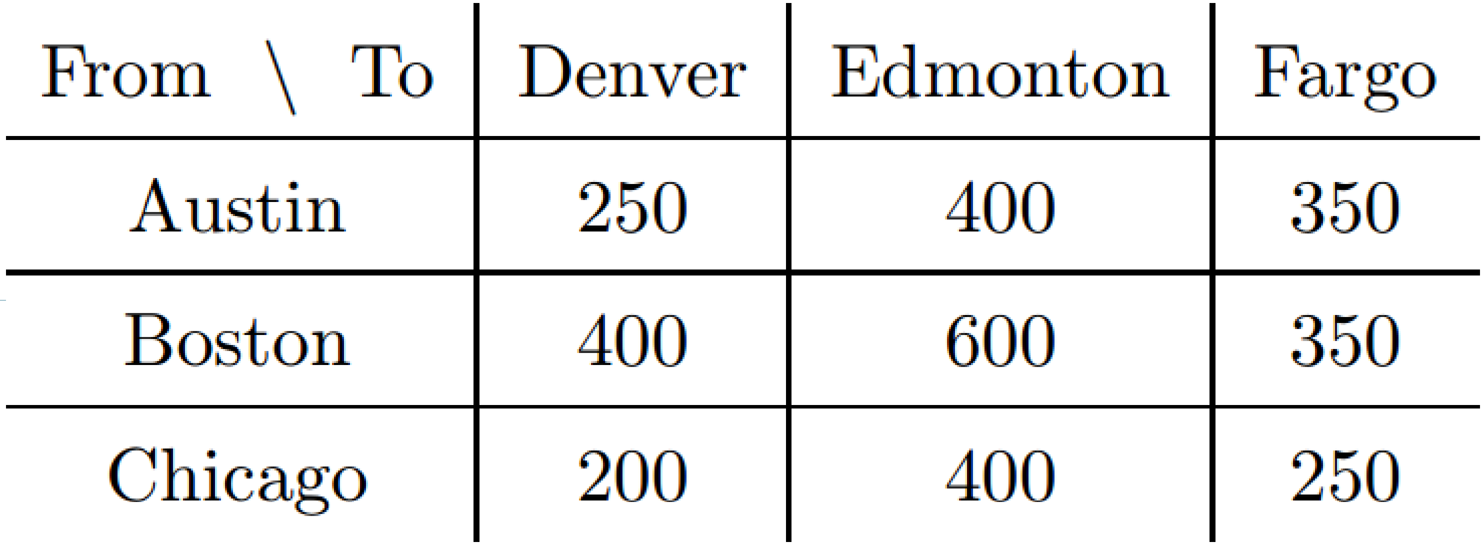
## Linear Programming (Algorithms)

| | | | |
|---|---|---|---|
| 1947 | George Dantzig | • | Primal Simplex |
| 1954 | Carlton Lemke | ★ | Dual Simplex |
| 1954 | G. Dantzig and W. Orchard Hays | • | Revised Simplex Method |
| 1958 | L. Ford and D. Fulkerson | • | Column Generation |
| 1979 | Leonid Khachiyan | • | Ellipsoid Method |
| 1984 | Narendra Karmarkar | ★ | Interior Point Method |

**Polynomial running time**

# Problem Definition- Salesperson assignment

Assume you have 3 salespersons in different cities and three other cities to which you would like them to visit with the below cost matrix of flights. To which city should I assign each salesperson to minimize cost?



| From \ To | Denver | Edmonton | Fargo |
|---|---|---|---|
| Austin | 250 | 400 | 350 |
| Boston | 400 | 600 | 350 |
| Chicago | 200 | 400 | 250 |

# Combinatorial Optimization

## The Assignment Problem

$$\text{(P)} \qquad \min \sum_{i\in V_1} \sum_{j\in V_2} c_{ij}x_{ij}$$

$$\text{s.t.} \qquad \sum_{i\in V_1} x_{ij} = 1 \qquad \forall j \in V_2$$

$$\sum_{j\in V_2} x_{ij} = 1 \qquad \forall i \in V_1$$

$$x_{ij} \in \{0,1\} \qquad \forall i \in I, j \in J$$

$V_1$ *denote the set of salesperson*

$V_2$ *denote the set of cities salespersons must visit.*

$c_{ij}$ *denote the cost of travelling for salesperson* $i \in V_1$ *to travel to city* $j \in V_2$*.*

$x_{ij}$ *denote whether salesperson* $i \in V_1$ *will travel to city* $j \in V_2$*.*

# Combinatorial Optimization

## The Assignment Problem

$$(\mathrm{P}) \qquad \min \sum_{i \in V_1} \sum_{j \in V_2} c_{ij} x_{ij}$$

**Objective Function- Linear**

$$\text{s.t.} \qquad \sum_{i \in V_1} x_{ij} = 1 \qquad \forall j \in V_2$$

$$\sum_{j \in V_2} x_{ij} = 1 \qquad \forall i \in V_1$$

**Constraints- Finite and Linear**

$$x_{ij} \in \{0, 1\} \qquad \forall i \in I, j \in J$$

**Variable Definition- Binary**

# Combinatorial Optimization

## The Assignment Problem

$$(P) \qquad \min \sum_{i \in V_1} \sum_{j \in V_2} c_{ij} x_{ij}$$

$$\text{s.t.} \qquad \sum_{i \in V_1} x_{ij} = 1 \qquad \forall j \in V_2$$

$$\sum_{j \in V_2} x_{ij} = 1 \qquad \forall i \in V_1$$

$$x_{ij} \in \{0,1\} \qquad \forall i \in I, j \in J$$

### Current state of the art solution method

- The Hungarian Method Harold Kuhn

# Combinatorial Optimization

## The Assignment Problem

### Interesting Result

$$
\begin{aligned}
(\mathrm{P}) \quad & \min \sum_{i\in V_1}\sum_{j\in V_2} c_{ij}x_{ij} \\
\text{s.t.} \quad & \sum_{i\in V_1} x_{ij} = 1 \qquad \forall j \in V_2 \\
& \sum_{j\in V_2} x_{ij} = 1 \qquad \forall i \in V_1 \\
& x_{ij} \in \{0,1\} \qquad \forall i \in I, j \in J
\end{aligned}
$$

$$=$$

$$
\begin{aligned}
(\bar{P}) \quad & \min \sum_{i\in V_1}\sum_{j\in V_2} c_{ij}x_{ij} \\
\text{s.t.} \quad & \sum_{i\in V_1} x_{ij} = 1 \qquad \forall j \in V_2 \\
& \sum_{j\in V_2} x_{ij} = 1 \qquad \forall i \in V_1 \\
& x_{ij} \in [0,1] \qquad \forall i \in V_1, j \in V_2
\end{aligned}
$$

# Solving the Assignment Problem

## Linear Programming (Algorithms)

| | | |
|---|---|---|
| 1947 | George Dantzig | Primal Simplex |
| 1954 | Carlton Lemke | Dual Simplex |
| 1954 | G. Dantzig and W. Orchard Hays | Revised Simplex Method |
| 1958 | L. Ford and D. Fulkerson | Column Generation |
| 1979 | Leonid Khachiyan | Ellipsoid Method |
| 1984 | Narendra Karmarkar | Interior Point Method |

# Problem Definition- Depot Location

There are 4 potential depot locations, each with a different installation cost, that must serve a set of 8 retail stores. Each depot has a restricted list of possible stores it can serve as below. What is the minimum cost of installing depots such that each store has at least one depot?

| Store | Depot 1 | Depot 2 | Depot 3 | Depot 4 |
|---|---|---|---|---|
| 1 | Yes | No | No | Yes |
| 2 | No | Yes | No | Yes |
| 3 | No | Yes | Yes | Yes |
| 4 | No | No | Yes | Yes |
| 5 | Yes | Yes | No | No |
| 6 | No | No | Yes | Yes |
| 7 | Yes | No | No | No |
| 8 | Yes | Yes | No | No |

# Combinatorial Optimization

## The Set Covering Problem

$$
\begin{aligned}
\text{(P)} \qquad & \min \sum_{j \in J} f_j x_j \\
\text{s.t.} \qquad & \sum_{j \in J} a_{ij} x_j \geq 1 \qquad \forall i \in I \\
& x_j \in \{0, 1\} \qquad \forall j \in J
\end{aligned}
$$

*$J$ denote the set of depots.*

*$f_j$ denote the cost of opening depot $j \in J$.*

*$I$ denote the set of retail stores.*

*$a_{ij}$ is 1 if depot $j \in J$ can serve store $i \in I$.*

*$x_j$ denote whether depot j is open.*

# Combinatorial Optimization

## The Set Covering Problem

$$
\begin{aligned}
\text{(P)} \qquad & \min \sum_{j \in J} f_j x_j \\
\text{s.t.} \qquad & \sum_{j \in J} a_{ij} x_j \geq 1 \qquad \forall i \in I \\
& x_j \in \{0, 1\} \qquad \forall j \in J
\end{aligned}
$$

**Objective Function- Linear**

**Constraints- Finite and Linear**

**Variable Definition- Binary**

# Combinatorial Optimization

## The Set Covering Problem

**Question**

$$
\begin{aligned}
(\mathrm{P}) \quad & \min \sum_{j \in J} f_j x_j \\
\text{s.t.} \quad & \sum_{j \in J} a_{ij} x_j \geq 1 \quad \forall i \in I \\
& x_j \in \{0,1\} \quad \forall j \in J
\end{aligned}
\quad \overset{?}{=} \quad
\begin{aligned}
(\mathrm{P}) \quad & \min \sum_{j \in J} f_j x_j \\
\text{s.t.} \quad & \sum_{j \in J} a_{ij} x_j \geq 1 \quad \forall i \in I \\
& x_j \in [0,1] \quad \forall j \in J
\end{aligned}
$$

# Combinatorial Optimization

## The Set Covering Problem

### Question

$$(P) \quad \min \sum_{j \in J} f_j x_j$$

$$\text{s.t.} \quad \sum_{j \in J} a_{ij} x_j \geq 1 \quad \forall i \in I$$

$$x_j \in \{0, 1\} \quad \forall j \in J$$

$$\neq$$

$$(P) \quad \min \sum_{j \in J} f_j x_j$$

$$\text{s.t.} \quad \sum_{j \in J} a_{ij} x_j \geq 1 \quad \forall i \in I$$

$$x_j \in [0, 1] \quad \forall j \in J$$

Cannot use LP Methods

Is NP Hard

# Combinatorial Optimization

## Question

What changed between

The Set Covering Problem & The Assignment Problem ?

Totally Unimodular Matrix

Total Unimodularity = Easy Problems

# Solving Combinatorial Problems

## Easy Problems

-Specialized Algorithms that run in Polynomial Time

-Find a complete formulation and solve the LP model

-Model may be polynomial in size.

-Model may have exponential number of inequalities but can be separated in polynomial time.

# Solving Combinatorial Problems

## Difficult Problems (NP hard)

-Approximation Algorithms

-Cutting Plane Methods

-Branch and Bound/ Branch and Cut  Most often used in practice

-Group Theoretic Approach  Seldom used in practice

# Problem Definition- A 24 hour restaurant

In a 24 hour restaurant they have established shifts for their full time workers to cover the day. However, demand for workers varies for each 3 hour time window. Due to the time of day, the wage for a full time worker in each shift varies. What is the minimum cost of employing workers to satisfy demand?

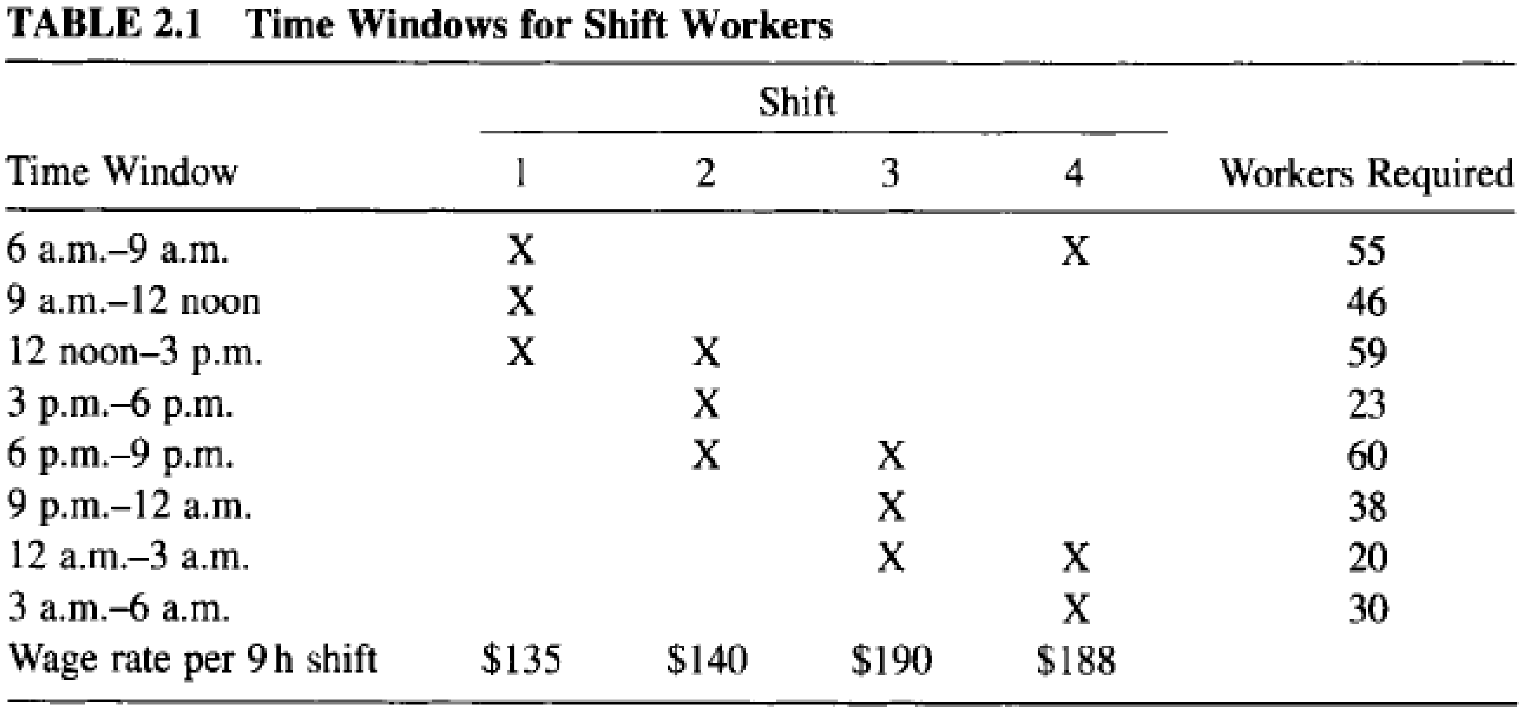


**TABLE 2.1** **Time Windows for Shift Workers**

| | Shift | | | | |
|---|---|---|---|---|---|
| Time Window | 1 | 2 | 3 | 4 | Workers Required |
| 6 a.m.–9 a.m. | X | | | X | 55 |
| 9 a.m.–12 noon | X | | | | 46 |
| 12 noon–3 p.m. | X | X | | | 59 |
| 3 p.m.–6 p.m. | | X | | | 23 |
| 6 p.m.–9 p.m. | | X | X | | 60 |
| 9 p.m.–12 a.m. | | | X | | 38 |
| 12 a.m.–3 a.m. | | | X | X | 20 |
| 3 a.m.–6 a.m. | | | | X | 30 |
| Wage rate per 9 h shift | $135 | $140 | $190 | $188 | |

# Integer Programming (Optimization)

## The Set Covering Problem

$$
\begin{aligned}
\text{(P)} \qquad & \min \sum_{j \in J} f_j x_j \\
\text{s.t.} \qquad & \sum_{j \in J} a_{jt} x_j \geq d_t \qquad \forall t \in T \\
& x_j \geq 0 \text{ and integer} \qquad \forall j \in J
\end{aligned}
$$

*$J$ denote the set of shifts.*

*$f_j$ denote the wage of an employee on that shift*

*$T$ denote the set of 3 hour time windows.*

*$d_t$ is the # of workers needed in time window t*

*$a_{jt}$ is 1 if shift $j \in J$ covers time window t$\in$ $T$.*

*$x_j$ denote the number of employees in shift $j \in J$ .*

# Solving Pure Integer Programs

-Approximation Algorithms

-Branch and Bound/ Branch and Cut

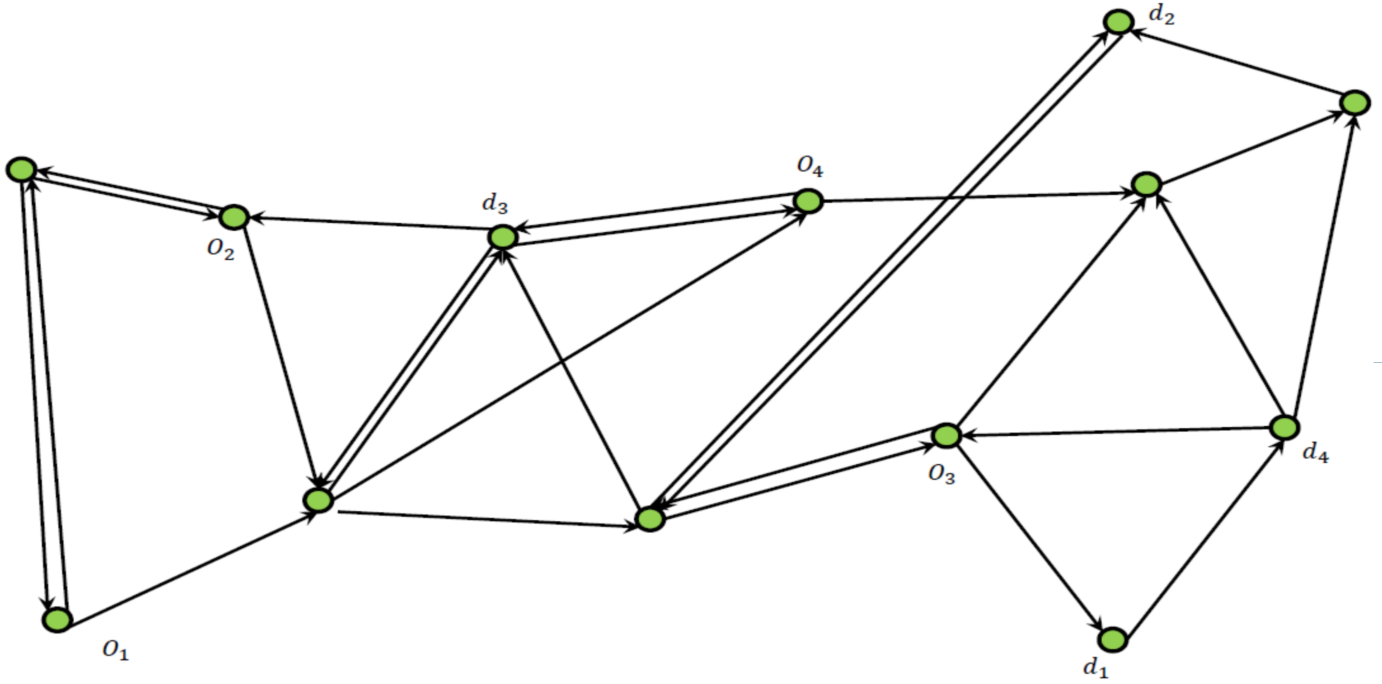
-Cutting Plane Methods
  -Gomory Cuts
  - Valid Inequalities
  - Lift and Project

-Decomposition Methods

-Group Theoretic Approach

# Problem Definition- Network Loading Problem

Suppose you are given a network and a set of requests with different origin and destination nodes and demand quantity that you have to route. For each link you can install a structure that allows a number u of units to pass. The cost of each structure varies per link. What is the minimum cost of installation to send all the requests?



# Problem Definition- Network Loading Problem

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**Notation**

$A$ *denote the set of arcs.*

$c_{ij}$ *is the cost of installing a structure on arc* $(i,j) \in A$

$u$ *is the flow capacity of a structure installed*

$N$ *denote the set of nodes*

$f_{ij}^{k}$ *is the portion of commodity k routed on arc* $(i,j) \in A$*.*

$x_{ij}$ *denotes the number of structures installed on arc* $(i,j) \in A$*.*

$K$ *denote the set of commodities.*

$\mathrm{o}(k)$ *is the origin of a commodity k.*

$\mathrm{d}(k)$ *is the destination of a commodity k.*

$W_k$ *is the demand quantity of a commodity k.*

# Mixed Integer Programming (Optimization)

$$(\mathrm{P}) \qquad \min \sum_{ij\in A} c_{ij} x_{ij}$$

$$\text{s.t.} \qquad \sum_{j\in N:(j,i)\in A} f^k_{ji} - \sum_{j\in N:(i,j)\in A} f^k_{ij} = \begin{cases} -1 & \text{if } i = o(k) \\ 0 & \text{if } i \notin \{o(k), d(k)\} \\ 1 & \text{if } i = d(k) \end{cases} \qquad \forall i \in N, \forall k \in K$$

$$\sum_{k\in K} W_k f^k_{ij} \leq u x_{ij} \qquad \forall ij \in A$$

$$f^k_{ij} \geq 0 \qquad \forall ij \in A, k \in K$$

$$x_{ij} \geq 0 \text{ and integer} \qquad \forall ij \in A$$

# Solving Mixed Integer Programs

-Approximation Algorithms

-Branch and Bound/ Branch and Cut

-Cutting Plane Methods

    -Mixed Integer Chvatal-Gomory Cuts

    - Mixed Integer Rounding Cuts

    - Disjunctive Cuts

-Decomposition Methods
    -Benders Decomposition

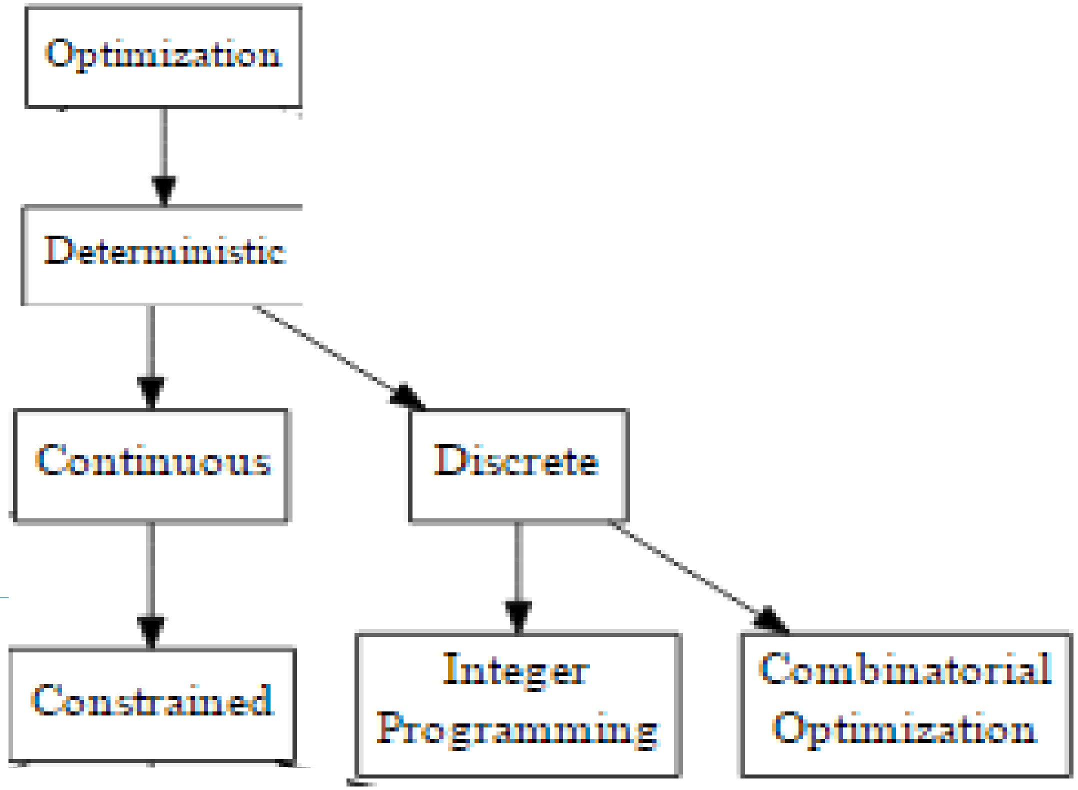
-Group Theoretic Approach

# Summary of today

- Focused on Linear objective functions and a finite set of linear constraints

| Variables | Classification | | Solution Methods |
|---|---|---|---|
| Continuous | Continuous Programming | | Simplex (Primal/Dual), Column Generation, Ellipsoid Method, Interior Point Methods |
| Binary | Combinatorial | Easy | Polynomial time Algorithms, LP methods with separation oracles |
| | | Hard | Implicit Enumeration, Cutting Planes, Group Theoretic Approach, Decomposition Methods |
| Integer | Integer Programming | | |
| Continuous & Integer | Mixed Integer Programming | | |

# Summary of today

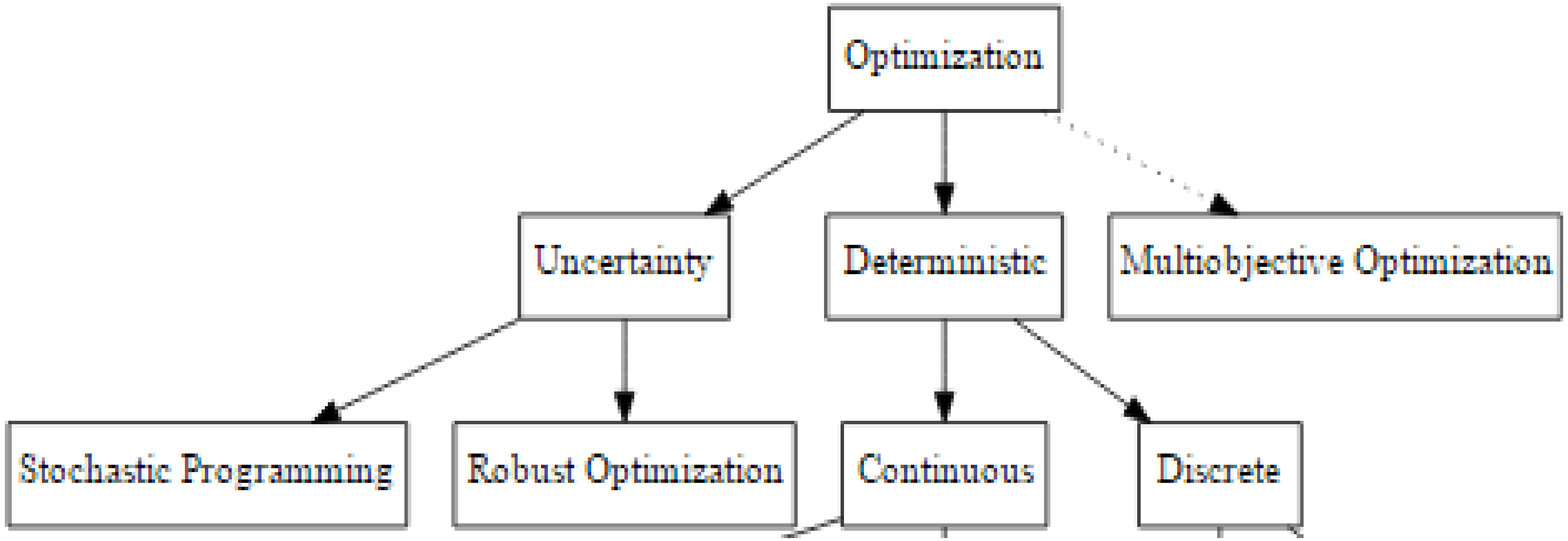


Source: http://www.neos-guide.org/content/optimization-taxonomy

# But there’s a lot more …

- Multiple Objective Functions
- Convex and Non Convex objective functions
- Convex and Non Convex constraints
- Infinite number of constraints
- Uncertain parameter data

# An Approximate Taxonomy (by the NEOS Guide)



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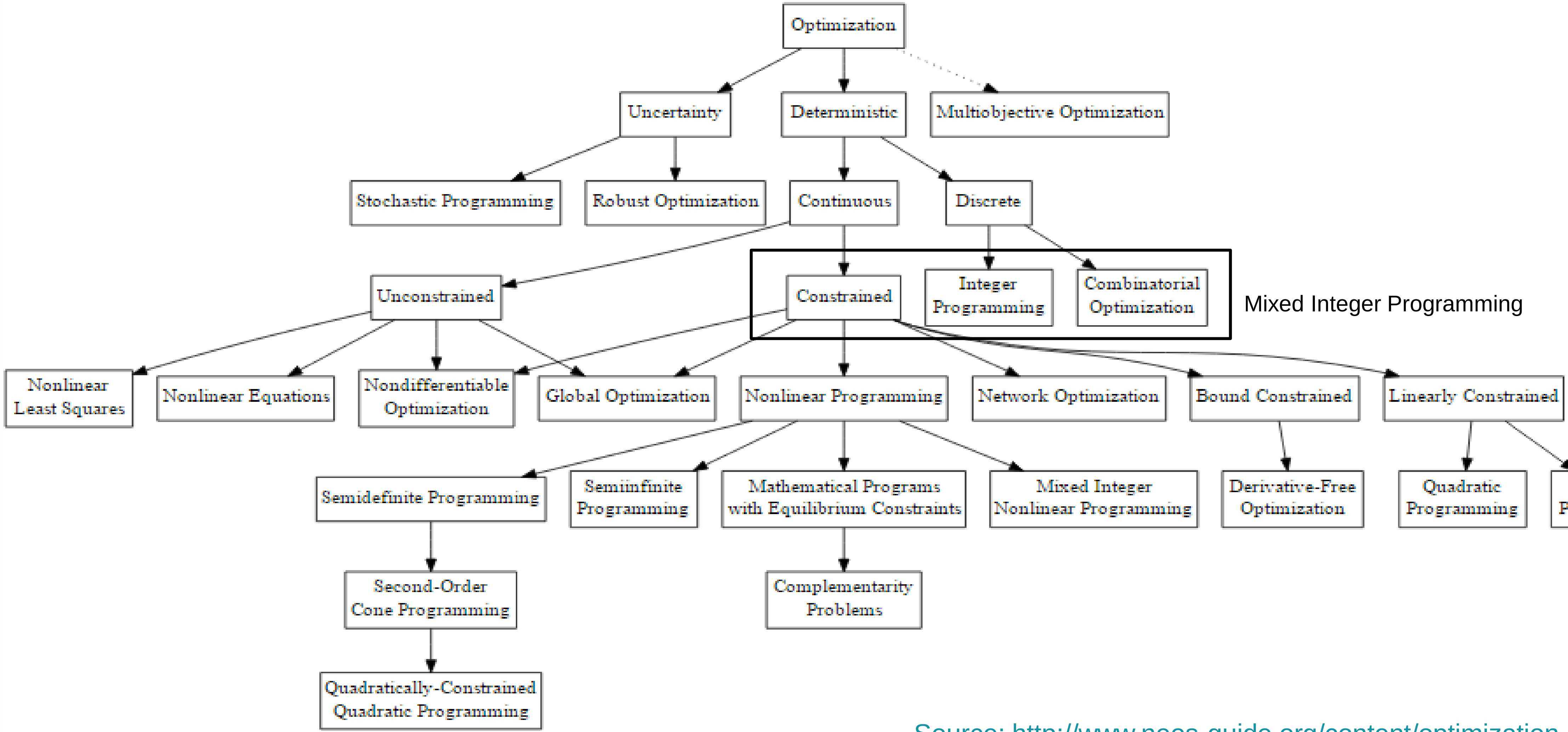

Mixed Integer Programming

Source: http://www.neos-guide.org/content/optimization-taxonomy

# The NEOS Server

- The NEOS Server is a free internet-based service for solving numerical optimization problems.
- Hosted by the Wisconsin Institutes of Discovery at the University of Wisconsin in Madison.
- Provides access to more than 60 state-of-the-art solvers, both commercial and open source, in more than a dozen optimization categories.
- Offers a variety of interfaces for accessing the solvers, and jobs run on distributed high-performance machines enabled by the HTCondor software.

Source: http://www.neos-guide.org/content/optimization-taxonomy

# Questions?

Don’t forget to sign up as a member @ www.mtl-students.com

Be a part of our team and add volunteer experience to your CCR

Email us at or.students.montreal@gmail.com or visit our website

Take advantage of our offer with Desjardins!

Thank you!