Formulations and Approximation Algorithms for Multi-level Facility Location Problems

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May 2016

Outline

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- ◇ Allocation of customers to open facilities

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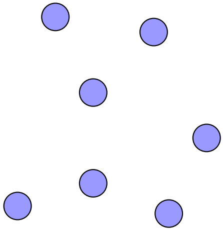
- Set of customers I
- Sets of potential facilities of levels 1 to k $(V_1, \cdots, V_k)$
- Setup costs f_{j_r} for each facility
- Allocation profits $c_{ij_1 \ldots j_k}$



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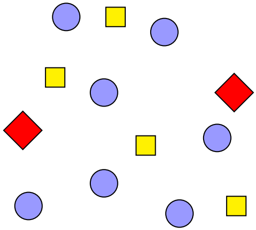

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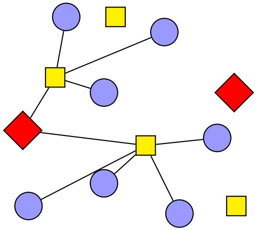

The MUFLP consists of selecting a set of facilities to open at each of the k levels and of assigning each customer to a set of facilities, exactly one at each level, while maximizing the difference of the total profit minus the setup cost for opening the facilities.

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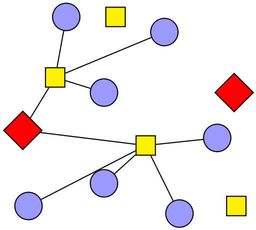

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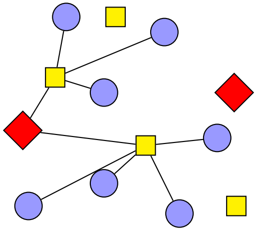

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The MpMP is a particular case of MUpLP when all f_{j_r} are zero.

Submodularity

Let N be a finite set and z be a real-valued function defined on the set of subsets of N and $\rho_e(W) = z(W \cup \{e\}) - z(W)$.

Definition

1. z is submodular if $\rho_e(W) \geq \rho_e(U), \quad \forall\, W \subseteq U \subseteq N$ and $e \in N \setminus U$.
2. z is nondecreasing if $\rho_e(W) \geq \rho_e(U) \geq 0, \quad \forall\, W \subseteq U \subseteq N$ and $e \in N$.

Some results for single level FLPs

- Nemhauser et al. (1978) presented a greedy heuristic for

$$\max_{S \subseteq N} \{z(S) : |S| \leq p \text{ and } z \text{ is submodular}\}. \tag{1}$$

Proposition

If the greedy heuristic is applied to problem (1) *then*
$$\frac{Z - Z^G}{Z - z(\emptyset)} \leq \frac{p-1}{p} \text{ and } \frac{Z - Z^G}{Z - z(\emptyset) + p\theta} \leq \left(\frac{p-1}{p}\right)^p.$$

where, Z is the optimal value, Z^G the value obtained by the greedy heuristic and $\rho_e(W) \geq \theta$ for all $W \subseteq N$ and $e \in N \backslash W$.

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When z is also nondecreasing, that is $\theta = 0$ (e.g. p-MP)

$$\frac{Z - Z^G}{Z - z(\emptyset)} \leq \left(\frac{p-1}{p}\right)^p \leq 1/e,$$

and the bound is tight.

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- MILP Formulations (Nemhauser and Wolsey, 1981)

A Submodular Representation for the MUpLP

$$\max_{R \subseteq V} \left\{ \sum_{i \in I} \max_{j_1 \in R_1, \cdots, j_k \in R_k} c_{ij_1 \cdots j_k} - \sum_{r=1}^{k} \sum_{j_r \in R_r} f_{j_r} : |R_r| \le p_r \right\}$$

The above objective function does not satisfy submodularity.

Example Submodularity

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Let Q be the set of all possible simple paths $(j_1, \cdots, j_k)$ and $N = Q \cup V$.

$$z(S, R) = h(S, R) + f(S, R) = \sum_{i \in I} \max_{(j_1, \cdots, j_k) \in S} c_{ij_1 \cdots j_k} - \sum_{r=1}^{k} \sum_{j \in R_r} f_{j_r}.$$

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$$\max_{(S,R) \subseteq N} \left\{ z(S,R) : |N_r(S)| \leq p_r \text{ and } N_r(S) = R_r \ \forall r \right\},$$

where $N_r(S) = \{j_r \in V_r : j_r \in s \text{ for some } s \in S\}$ and z satisfies submodularity. However, in general, z is not nondecreasing. (Ortiz-Astorquiza et al., 2015b)

The Greedy Heuristic for the MUpLP

Let $(S,R)^0 \leftarrow \emptyset$, $N^0 \leftarrow N$ and $t \leftarrow 1$

while $t < p_1 + 1$ **do**

Select $A_{q^*}(t) \subseteq N^{t-1}$ for which $\rho_{A_{q^*}(t)}((S,R)^{t-1}) = \max_{A_q(t) \in N^{t-1}} \rho_{A_q(t)}((S,R)^{t-1})$ with ties broken arbitrarily. Set $\rho_{t-1} \leftarrow \rho_{A_{q^*}(t)}((S,R)^{t-1})$

if $\rho_{t-1} \leq 0$ **then**

Stop with $(S,R)^{t-1}$ as the greedy solution

else

Set $(S,R)^t \leftarrow (S,R)^{t-1} \cup A_{q^*}(t)$ and $N^t \leftarrow N^{t-1} \setminus A_{q^*}(t)$

end if

for r such that $|N_r(S^t)| = p_r$ **do**

Set $N^t \leftarrow N^t \setminus \{q : \exists j_r \in V_r \setminus R_r^t\}$

end for

$t \leftarrow t + 1$

end while

Worst-case bounds for greedy heuristics

Under the assumption that $c_{ij_1 \cdots j_k} = c_{ij_1} + \cdots + c_{j_{k-1}j_k} \geq 0$.

Proposition

If the greedy heuristic for the MUpLP terminates after t^* iterations,

$$\frac{Z - Z^G}{Z - z(\emptyset) + p_1\theta} \leq \frac{t^*}{p_1}\left(\frac{p_1 - 1}{p_1}\right)^{p_1} \leq \left(\frac{p_1 - 1}{p_1}\right)^{p_1} \leq 1/e.$$

Proposition

If the greedy heuristic is applied to MpMP, then

$$\frac{H - H^G}{H} \leq \left(\frac{p_1 - 1}{p_1}\right)^{p_1} \leq 1/e,$$

and the bound is tight.

This work was submitted for publication (Ortiz-Astorquiza et al., 2015a)

A Submodular MILP Formulation

Let x_q be 1 if path $q \in Q$ is open and y_{j_r} be 1 if facility $j_r \in V_r$ is open, 0 otherwise. The MUpLP can be formulated as

$$
\begin{aligned}
\text{(SF)} \quad \max \quad & \eta - \sum_{r=1}^{k} \sum_{j_r \in V_r} f_{j_r} y_{j_r} \\
\text{s.t.} \quad & \eta \le h(S) + \sum_{q \in Q \setminus S} \rho_q(S) x_q \quad S \subseteq Q && (2) \\
& \sum_{q \in Q : j_r \in q} x_q \le M_r y_{j_r} \quad j_r \in V_r,\ r = 1, \cdots, k && (3) \\
& \sum_{j_r \in V_r} y_{j_r} \le p_r \qquad r = 1, \cdots, k && (4) \\
& x_q \in \{0, 1\} \qquad q \in Q && (5) \\
& y_{j_r} \in \{0, 1\} \qquad j_r \in V_r, \quad r = 1, \cdots, k, && (6)
\end{aligned}
$$

where, $M_r = \min\{p_1, |Q|/V_r\}$ are sufficiently large values for $r = 1, \cdots, k$.

A Submodular MILP Formulation

Constraints (2) can be written as

$$\eta^i \leq c_{iq_t} + \sum_{q \in Q} (c_{iq} - c_{iq_t})^+ x_q \quad i \in I, \quad t = 0, \cdots .|Q| - 1,$$

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And since we assumed that $c_{ij_1 \cdots j_k} = c_{ij_1} + \cdots + c_{j_{k-1} j_k} \geq 0$, we can add the valid cut

$$\sum_{q \in Q} x_q \leq p_1.$$

Computational Results

Using CPLEX 12.6.1 we compare the submodular formulations for the MpMP and for the MUpLP with three previously presented formulations. A Path-based formulation (PBF, Aardal et al., 1999), an Arc-based formulation (ABF, Aardal et al., 1996; Gabor and Ommeren, 2010) and a Flow-based formulation (FBF, Kratica et al., 2014).

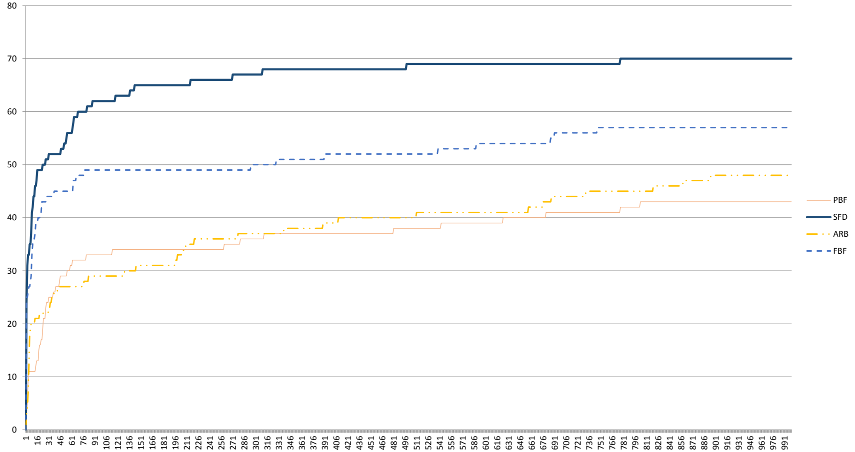
Other Formulations

	$k=2$	$k=3$	$k=4$	$\|I\|=500$	$\|I\|$=1,000	$\|I\|$=1,500	$\|I\|$=2,000	cap	Total	SGM sec	SGM nodes	Avg. %gap
SFD	36/39	22/25	12/12	16/16	37/37	12/16	2/4	21/21	70/76	3.46	13.44	1.24
FBF	28/39	18/25	11/12	13/16	33/37	6/16	2/4	21/21	57/76	10.54	13.70	3.19
ARB	28/39	14/25	6/12	13/16	27/37	4/16	1/4	19/21	48/76	46.91	0.14	0.01
PBF	35/39	8/25	0/12	8/16	24/37	6/16	2/4	15/21	43/76	-	-	-
Greedy	17/39	8/25	6/12	6/16	20/37	4/16	1/4	11/21	31/76	0.00	-	1.33

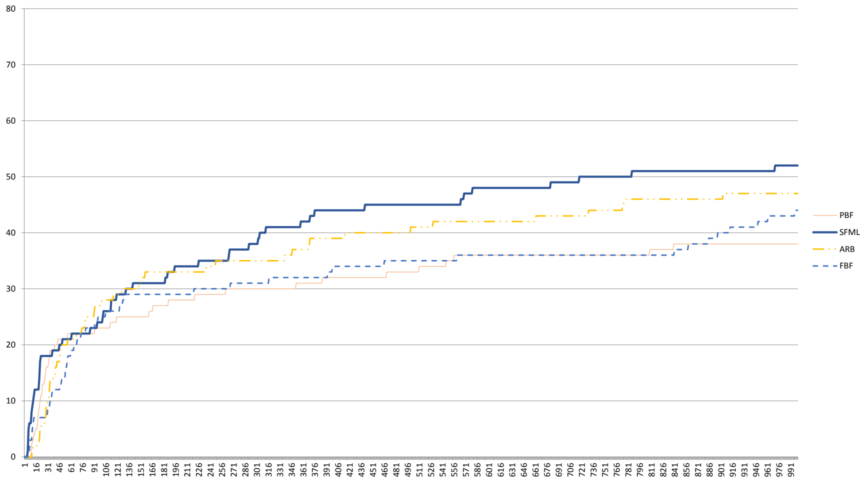
Table : Summary of the computational results for the MpMP.

	$k=2$	$k=3$	$k=4$	$\|I\|=500$	$\|I\|$=1,000	$\|I\|$=1,500	$\|I\|$=2000	cap	MUFLP	Total	SGM sec	SGM nodes	Avg. %gap
SFML	29/36	15/25	9/12	12/16	31/37	9/16	1/4	16/21	10/20	53/73	68.84	440.64	4.41
FBF	18/36	14/25	8/12	13/16	29/37	2/16	0/4	21/21	14/20	44/73	91.70	103.2	7.76
ARB	25/36	16/25	5/12	13/16	29/37	4/16	1/4	21/21	20/20	47/73	79.64	0.37	0.01
PBF	28/36	6/25	0/12	7/16	23/37	4/16	0/4	15/21	12/20	34/73	-	-	-
Greedy	1/36	0/25	0/12	0/16	1/37	0/16	0/4	0/21	0/20	2/73	0.00	-	5.88

Table : Summary of the Computational Results for the MUpLP.



Comparison of models by number of solved instances for MpMP



Comparison of models by number of solved instances for MUpLP

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Appendix

Greedy

Submodular Formulations

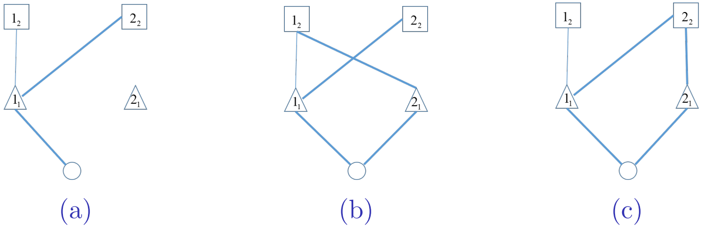
Example

c	1_2	2_2
1_1	1	1
2_1	100	1

(a) $T = \{1_1, 1_2, 2_2\}$ and $S = \{1_1, 2_2\}$ (b) $T \cup \{2_1\}$ and $S \cup \{2_1\}$

Vertex representation of the MUFLP

$v(T) = 1, v(T \cup \{2_1\}) = 100$ and $\rho_{2_1}(T) = 99$
$v(S) = 1, v(S \cup \{2_1\}) = 1$ and $\rho_{2_1}(S) = 0.$



Path-allocation representation of the MUFLP

(a) $T = \{(1_1, 1_2), (1_1, 2_2)\}$ and $S = \{(1_1, 2_2)\}$
(b) $T = \{(1_1, 1_2), (1_1, 2_2), (2_1, 1_2)\}$ and $S = \{(1_1, 2_2), (2_1, 1_2)\}$
(c) $T = \{(1_1, 1_2), (1_1, 2_2), (2_1, 2_2)\}$ and $S = \{(1_1, 2_2), (2_1, 2_2)\}$

back

Proposition

The greedy heuristic for the MUpLP can be executed in $O\left(p_1|V_1|\left(|V|\log|V| + E + |I|\right)\right)$ *time.*

Proof: At iteration t the subset $A_{q^*}(t) \subseteq N^{t-1}$ can be efficiently identified by solving a series of shortest path problems as follows. We consider the auxiliary directed graph $G^t = (V^t, A^t)$, where $A^t = \left\{(i,j) : i \in V_r^t, j \in V_{r+1}^t, r = 1, \ldots, k-1\right\}$. For each $a \in A^t$, we define its length as $w_{j_r j_{r+1}} = f_{j_{r+1}} - c_{j_r j_{r+1}}$ if $j_{r+1} \notin R^t$ and $w_{j_r j_{r+1}} = -c_{j_r j_{r+1}}$ if $j_{r+1} \in R^t$. This operation takes $O(|E|)$ time. We then compute a candidate path q, and its associated subset $A_q(t)$, associated with each facility $j \in V_1 \setminus R_1^t$ by solving a shortest path problem between j and all nodes in V_k. This can be done in $O(|V|\log|V| + |E|)$ time using the Fibonacci heap implementation of Dijkstra's algorithm (Ahuja et al., 1993). Finally, we evaluate $\rho_{A_q(t)}((S,R)^{t-1})$ for each candidate path q. This takes $O(|I|)$ time. Therefore, each iteration of the algorithm takes a total of $O(|V_1|\left(|V|\log|V| + E + |I|\right))$ time. Given that there are at most p_1 iterations in the algorithm, the result follows.

Property 1:

Under Assumption 1, there exists an optimal solution to the MUpLP in which every open facility at level r is assigned to exactly one facility at level $r+1$, for $r = 1, \ldots, k$ (i.e. coherent structure).

Property 2:

Under Assumption 1, there exists an optimal solution to the MUpLP in which at most p_1 paths are used, i.e. $|S| \leq p_1$.

back

Consider the polyhedron X defined as

$$\{(\eta, x, y_1, \cdots, y_k) : \eta \leq h(S) + \sum_{q \in Q \setminus S} \rho_q(S) x_q, \quad \forall S \subseteq Q,$$

$$x \in \{0,1\}^{|Q|}, \; y_r \in \{0,1\}^{|V_r|}, \; \eta \in \mathbb{R}\},$$

where the binary variables x_q can be interpreted as $x_q = 1$ if the path $q \in Q$ is open and 0 otherwise, and y_r corresponds to the incidence vector for each level r of the facilities that are open.

Proposition

Let $T \subseteq Q$, $N_r(T) \subseteq V_r$ for all r, and $(\eta, x^T, y_1^T, \cdots, y_k^T)$ where $x^T, y_1^T, \cdots, y_k^T$ are the incidence vectors of T and $N_r(T)$, respectively. Then, $(\eta, x^T, y_1^T, \cdots, y_k^T) \in X$ if and only if $\eta \leq h(T)$.

back

Also, note that since $h(S)$ is the sum of $|I|$ submodular set functions, one for each $i \in I$, we can replace the objective function η by $\sum_{i\in I} \eta^i$ and constraints (2) with

$$\eta^i \leq h^i(S) + \sum_{q\in Q\setminus S} \rho_q^i(S)x_q \quad i \in I,\ S \subseteq Q, \tag{7}$$

where $\rho_q^i(S) = h^i(S \cup \{q\}) - h^i(S)$. Moreover, most of these inequalities are redundant. First, note that for $S \subseteq Q$ and $i \in I$ given, the right-hand side of their associated constraint (7) does not change if the summation is taken over all $q \in Q$, since $\rho_q^i(S) = 0$ for $q \in S$. Also, $h^i(S) = c_{is_1,\cdots,s_k}$ for some $s_1, \cdots, s_k \in S$. For simplicity, we write c_{is} for $s \in S \subseteq Q$. Then, $\rho_q^i(S) = c_{iq} - c_{is}$ if $c_{iq} > c_{is}$ or $\rho_q^i(S) = 0$ if $c_{iq} \leq c_{is}$. For any S, its associated constraint (7) can thus be written as

$$\eta^i \leq c_{is} + \sum_{q\in Q} (c_{iq} - c_{is})^+ x_q,$$

for some $s \in S$ and $\chi^+ = \max\{0, \chi\}$. Therefore, if for each $i \in I$ we consider the ordering $0 = c_{iq_0} \leq c_{iq_1} \leq \cdots \leq c_{iq_{|Q|}}$, we may select only the sets $S_q = \{q\}$ with $q = q_0, \cdots, q_{|Q|-1}$ in constraints (7).

Proposition

The MpMP can be formulated as

$$\begin{aligned} \textit{maximize} \quad & \sum_{i \in I} \eta^i \\ \textit{subject to} \quad & (3) - (6) \\ & \eta^i \leq c_{iq_t} + \sum_{q \in Q} (c_{iq} - c_{iq_t})^+ x_q \quad i \in I, \quad t = 0, \cdots, |Q| - 1 \end{aligned} \tag{8}$$

Proof:
Since constraints (8) are a subset of constraints (2), we only need to show that if (ζ, x^T, y^T) does not satisfy constraints (2) (i.e $\zeta > h^{\hat{i}}(T)$ for some $\hat{i}$, by Proposition 3.2) for a given $T \subseteq Q$, then (ζ, x^T, y^T) is also infeasible with respect to constraints (8). Thus, suppose $h^{\hat{i}}(T) = \max_{q \in T} c_{iq} = c_{\hat{i}q_t}$, then the associated t^{th} inequality (8) would be

$$\zeta \leq c_{\hat{i}q_{t-1}} + \sum_{q \in Q} (c_{\hat{i}q} - c_{\hat{i}q_{t-1}})^+ x_q^T = c_{\hat{i}q_{t-1}} + c_{\hat{i}q_t} - c_{\hat{i}q_{t-1}} = c_{\hat{i}q_t} = h^{\hat{i}}(T),$$

which contradicts $\zeta > h^{\hat{i}}(T)$ and the result follows.

A Path-based Formulation

$$
\begin{aligned}
\text{(PBF)}\ \max \quad & \sum_{i\in I}\sum_{q\in Q} c_{iq}x_{iq} - \sum_{r=1}^{k}\sum_{j_r\in V_r} f_{j_r}y_{j_r} \\
\text{s. t.} \quad & \sum_{q\in Q} x_{iq} = 1 \qquad i\in I \\
& \sum_{q\in Q: j_r\in q} x_{iq} \le y_{j_r} \quad i\in I,\quad j_r\in V_r,\quad r=1,\cdots,k \\
& \sum_{j_r\in V_r} y_{j_r} \le p_r \qquad r=1,\cdots,k \\
& x_{iq}\ge 0 \qquad i\in I,\quad q\in Q \\
& y_{j_r}\in\{0,1\} \qquad j_r\in V_r,\quad r=1,\cdots,k.
\end{aligned}
$$

back

(Aardal et al., 1999)

An Arc-based Formulation

$$
\begin{aligned}
\text{(ABF) maximize} \quad & \sum_{i \in I} \sum_{j_1 \in V_1} c_{ij_1} x_{ij_1} + \sum_{i \in I} \sum_{r=1}^{k-1} \sum_{(a,b) \in V_r \times V_{r+1}} c_{ab} z_{iab} - \sum_{r=1}^{k} \sum_{j_r \in V_r} f_{j_r} \\
\text{subject to} \quad & \sum_{j_1 \in V_1} x_{ij_1} = 1 \qquad i \in I \\
& \sum_{b \in V_2} z_{iab} = x_{ia} \quad i \in I, \quad a \in V_1 \\
& \sum_{b \in V_{r+1}} z_{iab} = \sum_{b' \in V_{r-1}} z_{ib'a} \quad i \in I, \; a \in V_1, \; r = 2, \cdots, k-1 \\
& x_{ij_1} \le y_{j_1} \qquad i \in I, \quad j_1 \in V_1 \\
& \sum_{a \in V_{r-1}} z_{iab} \le y_b \qquad i \in I, \; b \in V_r \; r = 2, \cdots, k \\
& \sum_{j_r \in V_r} y_{j_r} \le p_r \qquad r = 1, \cdots, k \\
& x_{ij_1} \ge 0 \;, z_{iab} \ge 0 \quad i \in I, \quad j_1 \in V_1, \; (a,b) \in V_r \times V_{r+1} \\
& y_{j_r} \in \{0, 1\} \qquad j_r \in V_r, \quad r = 1, \cdots, k.
\end{aligned}
$$

(Aardal et al., 1996; Gabor and Ommeren, 2010)

A Flow-based Formulation

$$
\begin{aligned}
\text{(FBF)} \quad \text{maximize} \quad & \sum_{r=1}^{k} \sum_{a \in V_{r+1}} \sum_{b \in V_r} c_{ab} z_{abr} - \sum_{r=1}^{k} \sum_{j_r \in V_r} f_{j_r} y_{j_r} \\
\text{subject to} \quad & \sum_{b \in V_1} z_{ab0} = 1 \qquad a \in I \\
& \sum_{b \in V_{r-1}} z_{abr-1} = \sum_{b \in V_{r+1}} z_{bar} \quad a \in V_r, \quad r = 1, \cdots, k-1 \\
& z_{abr} \leq m y_b \qquad a \in V_{r+1},\ b \in V_r \quad r = 1, \cdots, k \\
& \sum_{j_r \in V_r} y_{j_r} \leq p_r \qquad r = 1, \cdots, k \\
& z_{ijr} \geq 0 \qquad i \in V_{r+1}, \quad j \in V_r, \ r = 0, \cdots, k \\
& y_{j_r} \in \{0, 1\} \qquad j_r \in V_r, \quad r = 1, \cdots, k.
\end{aligned}
$$

(Kratica et al., 2014)